

# CONTINUOUS HOMOTOPY INVARIANCE OF BIVARIANT LOCAL CYCLIC HOMOLOGY FOR $\sigma$ - $C^*$ -ALGEBRAS

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**ABSTRACT.** We establish the continuous homotopy invariance of bivariant local cyclic homology on the category of all  $\sigma$ - $C^*$ -algebras. The argument relies vitally on an isomorphism between the smooth and continuous cylinder constructions using a technical criterion due to Meyer. As a consequence we compute the local cyclic homology of the infinite sphere.

To any Fréchet algebra one can apply the *precompact bornology* functor  $\text{Cpt}(-)$ , which converts it into a complete bornological algebra. Then one can apply the dissection functor  $\text{diss}(-)$  to produce an ind-Banach algebra. Now the  $\mathcal{X}$ -complex formalism may be deployed to the (bornologically completed) analytic tensor algebra  $\mathcal{T}_{\text{an}}(-)$  to define the bivariant local cyclic homology of two Fréchet algebras  $A, B$  as

$$\text{HL}_*(A, B) = H_*^{\text{loc}}(\mathcal{X}(\mathcal{T}_{\text{an}}(P(A))), \mathcal{X}(\mathcal{T}_{\text{an}}(P(B)))).$$

Here  $P(-) := \text{diss} \circ \text{Cpt}(-)$  and for the details concerning the rest we refer the readers to [2]. This is how we are going to define the bivariant local cyclic homology on the category of all Fréchet algebras following Meyer. Although we are mainly interested in the bivariant local cyclic homology of  $\sigma$ - $C^*$ -algebras, we need to look at the theory for the larger class of all Fréchet algebras. This is because in the arguments below we need to consider both smooth and continuous cylinder constructions and the smooth cylinder construction takes us beyond the category of  $\sigma$ - $C^*$ -algebras (but keeps us within that of Fréchet algebras).

Let us recall that a *locally multiplicative ind-Banach* algebra is an ind-Banach algebra, which is an inductive system of Banach algebras; whereas an arbitrary ind-Banach algebra is merely a monoid object in the symmetric monoidal category of ind-Banach spaces. A Fréchet algebra  $A$  is called *locally multiplicative* if its associated ind-Banach algebra  $P(A)$  is locally multiplicative. It is known that bivariant local cyclic homology satisfies smooth homotopy invariance on the category of all ind-Banach algebras. This was a designing criterion for Puschnigg [5, 6] and in the setting of ind-Banach algebras this result is proved in Theorem 5.45 of [2]. It is further known that bivariant local cyclic homology satisfies continuous homotopy invariance on the category of all locally multiplicative Fréchet algebras (see Corollary 6.29 of *ibid.*). From Meyer's characterization of locally multiplicative Fréchet algebras one knows that such algebras cannot have elements with unbounded spectral radii. A  $\sigma$ - $C^*$ -algebra is also a Fréchet algebra. However, it contains many elements with unbounded spectral radii, unless it is actually a  $C^*$ -algebra. Therefore, such a  $\sigma$ - $C^*$ -algebra cannot be locally multiplicative, i.e., the existing result on the continuous homotopy invariance is not applicable to the category of all  $\sigma$ - $C^*$ -algebras. In this article we prove the continuous homotopy invariance of bivariant local cyclic homology on the category of all  $\sigma$ - $C^*$ -algebras.

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**Terminology and convention:** *In the sequel, by a Fréchet algebra (resp. a  $\sigma$ - $C^*$ -algebra) we mean a complete Hausdorff locally convex  $\mathbb{C}$ -algebra that can be written as a countable inverse limit of Banach algebras (resp.  $C^*$ -algebras). Such Fréchet algebras are called locally multiplicatively convex in the literature (see, for instance, [3]); the readers should not confuse them with the locally multiplicative Fréchet algebras described above. All bivariate local cyclic homological constructions are applied to Fréchet algebras after applying the functor  $P(-)$ , which is suppressed from the notation for brevity.*

Let  $A$  be any Fréchet algebra. A bounded subset  $T \subset \text{Cpt}(A)$  is called *power bounded* if  $T^\infty := \bigcup_{n=1}^\infty T^n$  is bounded and the spectral radius of  $T$  is, by definition, the infimum of the numbers  $r \in \mathbb{R}_{>0}$  for which  $r^{-1}T$  is power bounded. If no such  $r$  exists, then the spectral radius of  $T$  is set to  $\infty$ . Let  $(\mathbf{Ban}, \hat{\otimes}_\pi)$  denote the symmetric monoidal category of all Banach spaces with Grothendieck's projective tensor product  $\hat{\otimes}_\pi$ . Note that a partial algebra in  $\mathbf{Ban}$  is a triple  $(X, \mu, R)$ , where  $X, R$  are objects of  $\mathbf{Ban}$  and  $\mu : R \rightarrow X$  is a morphism therein; an injective map  $R \rightarrow X \hat{\otimes}_\pi X$  in  $\mathbf{Ban}$  is also a part of the data that is conveniently suppressed from the notation. Let  $f : X \rightarrow B$  be a linear map, where  $B$  is an ind-Banach algebra. The *curvature* of  $f$ , denoted by  $\omega_f : R \rightarrow B$ , is the difference of the two maps

$$R \xrightarrow{\mu} X \xrightarrow{f} B \quad \text{and} \quad R \rightarrow X \hat{\otimes}_\pi X \xrightarrow{f \hat{\otimes}_\pi f} B \hat{\otimes}_\pi B \xrightarrow{\text{mult}} B.$$

For a fixed bounded disk  $S \subset R$ , the linear map  $f : X \rightarrow P(A) := \text{diss} \circ \text{Cpt}(A)$  is said to have *small curvature* with respect to  $S$ , if the restriction of  $\omega_f$  to  $S$  has *spectral radius* strictly less than 1, where the *spectral radius* of a bounded map  $g : S \rightarrow P(A)$  is simply the spectral radius of the bounded subset  $g(S)$  inside  $\text{Cpt}(A)$ . The previous assertion uses the fact that  $f$  is a linear map into an ind-Banach algebra of the form  $P(A)$ , where  $A$  is a Fréchet algebra. Let  $M(S; X, A)$  be the set of linear maps  $X \rightarrow P(A)$  with small curvature with respect to  $S$  and  $H(S; X, A)$  be the set of smooth homotopy classes of such maps, where the homotopies must themselves be elements of  $M(S; X, C^\infty([0, 1], A))$ , i.e., have small curvature with respect to  $S$ . A homomorphism  $f : A \rightarrow B$  of ind-Banach algebras is called an *approximate smooth local homotopy equivalence* if  $\mathcal{T}_{\text{an}}(f) : \mathcal{T}_{\text{an}}(A) \rightarrow \mathcal{T}_{\text{an}}(B)$  is a smooth local homotopy equivalence (see section 6.1.2. of *ibid.*). Let us recall

**Theorem 1** (Meyer, Theorem 6.19. [2]). *An algebra homomorphism  $f : A \rightarrow B$  between two ind-Banach algebras is an approximate smooth local homotopy equivalence if and only if it induces isomorphisms  $H(S; X, A) \rightarrow H(S; X, B)$  for all partial algebras  $(X, \mu, R)$  with a fixed unit ball  $S \subset R$ .*

In the sequel we denote by  $\text{ev}_i^\infty : A^\infty[0, 1] := C^\infty([0, 1], A) \rightarrow A$  and  $\text{ev}_i : A[0, 1] := C([0, 1], A) \rightarrow A$ ,  $i = 0, 1$ , the smooth and continuous cylinder constructions respectively with their corresponding evaluation maps. Both cylinder constructions are nicely compatible with the functor  $P(-)$ , provided the input is a Fréchet algebra. Due to the nuclearity of  $C^\infty([0, 1])$  as a Fréchet space, one has  $A^\infty[0, 1] \cong C^\infty([0, 1]) \hat{\otimes}_\pi A$ . Observe that  $A[0, 1] = C([0, 1]) \hat{\otimes}_{\sigma-C^*} A$  (maximal or minimal  $\sigma$ - $C^*$ -tensor product) and  $A^\infty[0, 1]$  are both Fréchet algebras themselves (see, for instance, Sections 1 and 2 of [4]). Recall that an algebra homomorphism  $f$  between ind-Banach algebras is called an *HL-equivalence* if  $\mathcal{X}(\mathcal{T}_{\text{an}}(f))$  is a local homotopy equivalence.

**Proposition 2.** *For any  $\sigma$ - $C^*$ -algebra  $A$ , the canonical Fréchet algebra homomorphism between the cylinder constructions  $\iota : A^\infty[0, 1] \rightarrow A[0, 1]$  is an HL-equivalence.*

*Proof.* By Theorem 6.11. of [2] it suffices to show that  $\iota$  is an approximate smooth local homotopy equivalence. The above Theorem 1 of Meyer gives us a criterion to ascertain that, which we now verify.

Let  $(X, \mu, R)$  be any partial algebra in  $\mathbf{Ban}$ . For the surjectivity of the induced map  $\iota : H(S; X, A^\infty[0, 1]) \rightarrow H(S; X, A[0, 1])$  we need to find for any  $h \in M(S; X, A[0, 1])$  an  $h' \in M(S; X, A^\infty[0, 1])$ , such that  $\iota h'$  is homotopic to  $h$  via a smooth homotopy of small curvature. Let us choose a sequence of smooth kernels  $K_n(x, y)$  converging to  $\delta(x - y)$ . Using this sequence we produce via kernel smoothing a sequence of continuous linear maps (see, for instance, chapter 40 of [7])

$$\begin{aligned} \sigma_n : C([0, 1]) &\rightarrow C^\infty([0, 1]) \subset C([0, 1]) \\ f(x) &\mapsto [x \mapsto \int K_n(x, y)f(y)dy]. \end{aligned}$$

One extends them to linear maps  $\sigma_n = \sigma_n \otimes \text{id} : C([0, 1]) \otimes A \rightarrow C^\infty([0, 1]) \otimes A$ , where  $\otimes$  denotes the algebraic tensor product. This sequence can be further extended to a sequence of continuous linear maps  $\sigma_n : A[0, 1] \rightarrow A^\infty[0, 1]$  by density arguments, such that  $\iota \sigma_n : A[0, 1] \rightarrow A[0, 1]$  uniformly converges towards  $\text{id}$ . This implies that  $\iota \sigma_n h$  uniformly converges towards  $h$ , whence  $\omega_{\iota \sigma_n h}$  uniformly converges towards  $\omega_h$  (for a judicious choice of the approximation scheme via smooth kernels). Consequently, the spectral radius of  $\omega_{\iota \sigma_n h}$  restricted to  $S$  converges to that of  $\omega_h$  restricted to  $S$ , which is strictly less than 1 by hypothesis. The spectral radius of  $\omega_{\sigma_n h} : R \rightarrow A^\infty[0, 1]$  (resp.  $\omega_{\iota \sigma_n h} : R \rightarrow A[0, 1]$ ) restricted to  $S$  is that of the image  $\omega_{\sigma_n h}(S)$  (resp.  $\omega_{\iota \sigma_n h}(S)$ ) inside the bornological algebra  $\text{Cpt}(A^\infty[0, 1])$  (resp.  $\text{Cpt}(A[0, 1])$ ).

Now we show that the spectral radius of  $\omega_{\sigma_n h}(S)$  is strictly less than 1 for sufficiently large  $n$ . Let us set  $T = \omega_{\sigma_n h}(S)$  and  $T_r = r^{-1}T$ . Suppose that  $T_r^\infty$  viewed as a subset of  $\text{Cpt}(A[0, 1])$  via  $\iota$  is bounded, i.e.,  $\overline{T_r^\infty}$  is compact. Then the closure of  $T_r^\infty$  inside  $A^\infty[0, 1]$  is also compact, i.e.,  $T_r^\infty$  viewed as a subset of  $\text{Cpt}(A^\infty[0, 1])$  is bounded. Indeed, by the metrizable topology of  $A^\infty[0, 1]$  it suffices to check sequential compactness. Given any sequence of elements  $\{a_n\}$  in the closure of  $T_r^\infty$  inside  $A^\infty[0, 1]$ , we view them as a sequence in  $\overline{T_r^\infty} \subset A[0, 1]$  and by its compactness extract a convergent subsequence. The same subsequence serves as a convergent subsequence of  $\{a_n\}$  in the closure of  $T_r^\infty$  inside  $A^\infty[0, 1]$ . This analysis yields that the spectral radius of  $\omega_{\sigma_n h}(S) \subset \text{Cpt}(A^\infty[0, 1])$  is equal to that of  $\omega_{\iota \sigma_n h}(S) \subset \text{Cpt}(A[0, 1])$ . Since the latter converges to a number strictly less than 1, our assertion follows.

There is an affine smooth homotopy  $H_n : X \rightarrow (A[0, 1])^\infty[0, 1]$ , where  $H_n(t) = (1 - t)h + t\iota \sigma_n h$ , connecting the two linear maps  $h$  and  $\iota \sigma_n h$ . In the following paragraph we show that for sufficiently large  $n$  this affine homotopy  $H_n$  also has small curvature, which is equivalent to the assertion that the spectral radius of  $\omega_{H_n}(S)$  is strictly less than 1 for sufficiently large  $n$ . This will show that for a sufficiently large  $n$ , the element  $\sigma_n h \in M(S; X, A^\infty[0, 1])$  is a legitimate candidate for  $h'$ , whence the induced map  $\iota : H(S; X, A^\infty[0, 1]) \rightarrow H(S; X, A[0, 1])$  must be surjective.

Since  $\omega_{H_n}(S)$  is a precompact subset of the Fréchet algebra  $(A[0, 1])^\infty[0, 1]$ , its closure is compact. Set  $X_n = \omega_{H_n}(S) \subset C^\infty([0, 1], A[0, 1])$  and set  $X_n(t)$  to be the image of  $X_n$  after composition with the evaluation map  $\text{ev}_t^\infty : C^\infty([0, 1], A[0, 1]) \rightarrow A[0, 1]$  for  $t \in [0, 1]$ . We know that  $X_n(0) = \omega_h(S)$  (resp.  $X_n(1) = \omega_{\iota \sigma_n h}(S)$ ), whose spectral radius is  $R_0 < 1$  (resp.

$R_1 < 1$ ). We write the  $\sigma$ - $C^*$ -algebra  $A[0, 1]$  explicitly as a countable inverse limit

$$A[0, 1] \cong \varprojlim_{l \in \mathbb{N}} B_l,$$

where each  $B_l$  is a Banach algebra (actually a  $C^*$ -algebra), and we denote by  $\pi_l : \varprojlim_l B_l \rightarrow B_l$  the canonical projection homomorphisms. Let us introduce the following notation:

$$\pi_l(X_n) = V_l \text{ and } \pi_l(X_n(t)) = V(t)_l,$$

where each  $V(t)_l$  is a bounded subset of  $B_l$  for every  $l \in \mathbb{N}$  and  $\overline{(R_i^{-1}(V(i)_l))}^\infty$  is compact for  $i = 0, 1$ . This means that there are natural numbers  $N_l$  such that  $(R_0^{-1}(V(0)_l))^{N_l} \subset Q_l$ , where  $Q_l \subset B_l$  denotes the unit ball. By choosing  $n$  sufficiently large, we may arrange that  $(r^{-1}(V(t)_l))^{N_l} \subset Q_l$  for all  $t \in [0, 1]$  and for some  $r < 1$ . This is achieved as follows: due to the Heine–Borel property of Fréchet spaces, the compact subset  $\overline{(R_0^{-1}(X_n(0)))}^\infty \subset A[0, 1]$  is contained in a bounded open convex ball  $\mathcal{B}$  with respect to the standard Fréchet metric. We may choose  $\mathcal{B}$  small enough, so that  $(r^{-1}(\pi_l(\mathcal{B})))^{N_l} \subset Q_l$  for some  $r$  in the range  $\max\{R_0, R_1\} < r < 1$ . Now using the uniform convergence of  $X_n(1) = \omega_{\iota \sigma_n h}(S)$  towards  $X_n(0) = \omega_h(S)$  we can ensure that  $\overline{(R_1^{-1}(X_n(1)))}^\infty \subset \mathcal{B}$  for sufficiently large  $n$ . Observe that

$$(1) \quad \overline{(r^{-1}(X_n))}^\infty \cong \varprojlim_l \overline{(r^{-1}(V_l))}^\infty = \varprojlim_l \overline{(\cup_{p=1}^{N_l-1} (r^{-1}(V_l))^p) \cup ((r^{-1}(V_l))^{N_l})}^\infty.$$

Since  $Q_l^2 \subset Q_l$ ,  $\overline{Q_l}$  is compact for each  $l$  and the inverse limit of compact subsets is again a compact subset, we conclude that  $\varprojlim_l \overline{((r^{-1}(V_l))^{N_l})}^\infty \subset \varprojlim_l \overline{Q_l}$  is compact. Since  $r^{-1}(V_l)$  is precompact for each  $l$ , so is  $\cup_{p=1}^{N_l-1} (r^{-1}(V_l))^p$ . It follows that  $\varprojlim_l \overline{(\cup_{p=1}^{N_l-1} (r^{-1}(V_l))^p)}$  is compact, which implies that  $\overline{(r^{-1}(X_n))}^\infty$  is compact (see Equation (1)). Consequently  $r^{-1}(\omega_{H_n}(S)) = r^{-1}(X_n)$  is power bounded for some  $r < 1$ , whence the spectral radius of  $\omega_{H_n}(S)$  is strictly less than 1.

For injectivity, we need to show that whenever  $\iota h_0$  and  $\iota h_1$  are homotopic via a homotopy  $H \in M(S; X, (A[0, 1])^\infty[0, 1])$ , so are  $h_0$  and  $h_1$  via a homotopy in  $M(S; X, (A^\infty[0, 1])^\infty[0, 1])$ . Adopting a similar strategy as above, we can construct a sequence of smooth homotopies  $H'_n \in M(S; X, (A^\infty[0, 1])^\infty[0, 1])$ , such that  $\iota H'_n$  converges uniformly to  $H$ . It follows that  $\text{ev}_0^\infty H'_n$  and  $\text{ev}_1^\infty H'_n$  have the same class in  $H(S; X, A^\infty[0, 1])$ . Once again, for  $i = 0, 1$ , there are affine homotopies  $G_n : X \rightarrow (A^\infty[0, 1])^\infty[0, 1]$ , where  $G_n(t) = (1 - t)h_i + t\text{ev}_i^\infty H'_n$ . It suffices to show that  $G_n$  has small curvature for sufficiently large  $n$ , which would imply that  $\text{ev}_i^\infty H'_n$  (for sufficiently large  $n$ ) and  $h_i$ , for  $i = 0, 1$ , have the same class in  $H(S; X, A^\infty[0, 1])$ . Now using arguments as above, we can show that for sufficiently large  $n$ , the spectral radius of  $\omega_{G_n}(S)$  inside  $\text{Cpt}((A^\infty[0, 1])^\infty[0, 1])$  is strictly less than 1.  $\square$

**Theorem 3.** *Bivariant local cyclic homology satisfies continuous homotopy invariance on the category of all  $\sigma$ - $C^*$ -algebras, i.e., for any two  $\sigma$ - $C^*$ -algebras  $A, B$  the functors  $\text{HL}_*(B, -)$  and  $\text{HL}_*(-, B)$  send the evaluation maps  $\text{ev}_i : A[0, 1] \rightarrow A$  for  $i = 0, 1$  to isomorphisms.*

*Proof.* Consider the following commutative diagram of Fréchet algebras

$$\begin{array}{ccc}
& & A \\
& \nearrow \text{ev}_0^\infty & \uparrow \text{ev}_0 \\
A^\infty[0, 1] & \xrightarrow{\iota} & A[0, 1] \\
& \searrow \text{ev}_1^\infty & \downarrow \text{ev}_1 \\
& & A,
\end{array}$$

where  $\text{ev}_i^\infty$  (resp.  $\text{ev}_i$ ) are the evaluation maps from the smooth (resp. continuous) cylinder object. Due to the smooth homotopy invariance of bivariant local cyclic homology, we conclude that  $\text{ev}_i^\infty$  are mapped to isomorphisms by  $\text{HL}_*(B, -)$  and  $\text{HL}_*(-, B)$ . The above Proposition shows that  $\iota$  is an HL-equivalence, whence it is mapped to an isomorphism by the functors  $\text{HL}_*(B, -)$  and  $\text{HL}_*(-, B)$ . The assertion now follows from the two-out-of-three property of isomorphisms.  $\square$

**Corollary 4.** *Owing to the contractibility of the infinite sphere  $S^\infty := \varinjlim_n S^{2n+1}$ , we deduce*

$$\text{HL}_0(\mathbb{C}, \mathbb{C}(S^\infty)) \simeq \mathbb{C} \quad \text{and} \quad \text{HL}_1(\mathbb{C}, \mathbb{C}(S^\infty)) \simeq \{0\}.$$

**Corollary 5.** *We have established that bivariant local cyclic homology satisfies property (E1) of Cuntz [1] on the category of separable  $\sigma$ - $C^*$ -algebras.*

**Remark 6.** *The incompatibility of the cylinder constructions and the functor  $P(-)$  is one of the serious impediments to extending the above arguments beyond  $\sigma$ - $C^*$ -algebras. In fact, the continuous homotopy invariance of bivariant local cyclic homology may very well fail to hold for arbitrary pro  $C^*$ -algebras.*

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## REFERENCES

- [1] J. Cuntz. A general construction of bivariant  $K$ -theories on the category of  $C^*$ -algebras. In *Operator algebras and operator theory (Shanghai, 1997)*, volume 228 of *Contemp. Math.*, pages 31–43. Amer. Math. Soc., Providence, RI, 1998.
- [2] R. Meyer. *Local and analytic cyclic homology*, volume 3 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2007.
- [3] E. A. Michael. Locally multiplicatively-convex topological algebras. *Mem. Amer. Math. Soc.*, 1952(11):79, 1952.
- [4] N. C. Phillips.  $K$ -theory for Fréchet algebras. *Internat. J. Math.*, 2(1):77–129, 1991.
- [5] M. Puschnigg. Excision in cyclic homology theories. *Invent. Math.*, 143(2):249–323, 2001.
- [6] M. Puschnigg. Diffeotopy functors of ind-algebras and local cyclic cohomology. *Doc. Math.*, 8:143–245, 2003.
- [7] F. Trèves. *Topological vector spaces, distributions and kernels*. Academic Press, New York, 1967.

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